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Solipsism and philosophy of mathematics: intuitionists compared

Devoted to the memory of Anne Sjerp Troelstra

Abstract

This paper will consider L. E. J. Brouwer, A. Heyting and G. F. C. Griss as the first generation of Dutch intuitionists to look at the interrelationship between solipsism and mathematics.

In particular, our focus will be on Heyting, on the basis of the existence of some unpublished material (and also some difficult to find published material) revealing of the author's opinion on the subject and hence worthy of attention.

Keywords

foundations of mathematics, intuitionism, ontology, logic, Brouwer, Luitzen Egbertus Jan (1881–1966), Heyting Arendt (1898–1980)

1. L. E. J. Brouwer

In his 1905 booklet *Life, Art and Mysticism* (*Leven, kunst en mystiek*) Brouwer expresses a Hindu/mystical background:

> […] having contemplated the sadness of this world look into yourself. In you there is a consciousness, a consciousness which continually changes its content. Are you master of these changes? You say no, for you find yourself placed in a world which you have not created yourself […] what this self is we cannot further say […]. (1905: 400)

Life is suffering because it is a deviation from the inner self. Namely,

> Turning-into-oneself requires an effort […]. If, however, you succeed in overcoming all inertia and proceed, you will find that passions will be silenced, you will feel dead to the old world of perception, of time and space, and all other forms of plurality; and your eyes, no longer blindfolded, will be opened to a scene of joyful quiescence.
> When all images have been removed from the soul, and she beholds the Only One, then the naked essence of the soul finds the naked formless Essence of Divine Unity, the presence of the Superior Being waiting in the self. (Meister Eckhart)
> In God's wisdom it has been ordained that man must part from what is dearest to him.

Still, God himself wanted man to leave his inner self behind. Therefore we have to accept our destiny and simply "try to keep ourselves as far as possible from what is outside our 'self.'"

> Knowing this, you become reconciled with the erring world and accept its disconsolateness as natural; moreover, you feel it to be your inescapable karma, to which you have reconciled yourself and which you must fulfill, to see yourself driven away from the self, placed in life where pain and

> labor, desire and fear are your share and where all truth is veiled. You look on this life as the direction of your duty, and you live it as directed from within the self: in other words you recognize that all these earthly bonds remain your inevitable karma until God releases you. No new desires will be able to deflect you from your path and you will not want only increase the burden of your karma.

In this booklet Brouwer appears to be identifying the source of man's suffering as the self – outside world separation (which generates man's desire for power and its painful frustration) and suggesting that such suffering should not be augmented. By contrast, man should attempt to remain locked within the inner-self whilst remaining aware that this is not entirely possible because our involvement in the world has been designed by God (it is our burden, our inevitable karma). It might be said that Brouwer's 1905 booklet sees solipsism as the best way out for man, leading to a soothing "nothingness."

Brouwer could not incorporate this framework (see van Stigt 1990) but only its mathematical consequences into his 1907 dissertation *On the Foundation of Mathematics* (*Over de grondslagen der wiskunde*), i.e. mathematics must be developed a-linguistically, purposelessly, as an inner experience based on our aprioristic rudimentary intuition of time, so that it can be developed *within* man:

> To exist in mathematics means: to be constructed by intuition. (Brouwer 1975 [hereafter CW I]: 96)

> Mathematics is created by a free action independent of experience; it develops from a single aprioristic basic intuition, which may be called invariance in change as well as unity in multitude. (CW I: 97)

> A logical construction of mathematics, independent of the mathematical intuition is impossible – for by this method no more is obtained

> than a linguistic structure, which irrevocably remains separated from mathematics. (CW I: 97)

The aim of his dissertation is to highlight what he calls *a priori*, i.e. the only thing common to all mathematics and also sufficient to build up all mathematics. He states that it is the "intuition of the two--oneness" or "intuition of two-ity" and, since it is via this intuition that we become aware of time as change *per se*, the following can be said: "the only a priori element in science is time" (CW I: 61).

In a footnote at the end of the thesis, Brouwer specifies that there are "*two* discrete things thought together" (not just one) in the first act of natural number construction, because unity presupposes a perception of *two-ity*: "only afterwards this simplest mathematical system is projected on the first thing and *the ego which thinks the thing*" (CW I: 97).

In other words, the two-ity intuition is an *a priori* not only of our mathematical knowledge but also of our general knowledge, i.e. it is an *epistemological condition* because it allows subject to be distinguished from non-subject (an object thought). Furthermore, the regularity of a mathematical system of this sort is projected onto life in order to generate causal systems. Man obliges regularity to be projected onto nature by excluding observations which would disturb regularity: "he desires this regularity, because it strengthens him in the struggle for life, rendering him capable of predicting, and taking action" (CW I: 53). The possibility that this space could be *a priori* is ruled out (CW I: 68–69)[1].

[1] This is strongly re-affirmed in his 1952 paper *Historical Background, Principles and Methods of Intuitionism*: "for space the observational standpoint became untenable when, in the course of the 19th and the beginning of the 20th century, as a consequence of a series of discoveries with which the names of Lobatchefsky, Bolyai, Riemann, Cayley, Klein, Hilbert, Einstein, Levi-Civita and Hahn are associated, mathematics was gradually transformed into a mere science of numbers." Furthermore, "simultaneously, besides observational space, a great number of

It was at the 1948 conference (Consciousness, Philosophy, and Mathematics) that Brouwer set out the steps from the inner self to the sciences in greater depth:

> Consciousness in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation. And it seems that only the status of sensation allows the initial phenomenon of the said transition. The initial phenomenon is a *move of time*. By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains the former one as a past sensation. […] through this distinction between present and past, [consciousness] recedes from both and from stillness, and becomes *mind*. As mind it takes the function of a subject experiencing the present as well as the past sensation as object. […] In measure of this estrangement, mind becomes disposed to desire and apprehension. […] The free-will phenomenon of causal attention occurs. It performs identifications of different sensations […] and in this way creates *iterative complexes of sensations*. An iterative complex of sensations, whose elements have an invariable order of succession in time, whilst if one of its elements occurs, all following elements are expected to occur likewise, in the right order of succession, is called a *causal sequence*. There are iterative complexes of sensations whose elements are permutable in point of time. Some of them are completely estranged from the subject. They are called *things*. For instance *individuals*, i.e. human bodies, the home body of the subject included, are things. […] The whole of egoic sensations indissolubly connected with an individual is called the *soul*. […] Causal attention allows the development of the conative activity of the subject from spontaneous effort to forethinking enterprise by means of the free-will-phenomenon of *cunning act*. […] Mind, once

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other spaces, sometimes exclusively originating from logical speculations, with properties distinct from the traditional but no less beautiful, gradually found an arithmetical representation" (CW I: 508).

> having taken to causal attention, remains in a lasting *causal tension* [...]. (CW I: 481–482)

Here he introduces a distinction between simple consciousness and mind. Consciousness encompasses time perceptions whilst mind is a consciousness that separates itself off as a subject from the "world." This separation sparks desires and thus a new way of seeing the world is generated: causal attention, i.e. the act of pointing out sensation complexes. It is thus that things, individuals and also causal sequences are *shaped*. He argues that there can be no proof of the existence of other minds (CW I: 483):

> It is not unreasonable to derive this behaviour [the behaviour of individuals in general] from 'reason'. But unreasonable to derive it from 'mind'. For by the choice of this term the subject in its scientific thinking is induced to place in each individual a mind with free-will dependent on this individual, thus elevating itself to a mind of second order experiencing incognizable alien consciousness as sensations. Quod non est. And which moreover would have the consequence that the mind of second order would causally think about the pluralified mind of first order, then cooperatively study the science of the pluralified mind, and in consequence of this study assign a mind of second order with sensation of alien consciousness to other individuals, thus once more elevating itself, this time to a mind of third order. And so on. Usque ad infinitum.

For the purposes of our comparison it is important to stress Brouwer's *list of steps* from consciousness to objects:

- consciousness;
- time;
- mind;
- causality;
- objects and living beings;

– the fact that space is excluded as *a priori* (CW I: 68–69) and *a negative attitude* to these steps.

At the end of his paper, Brouwer adds:

> In default of a plurality of mind, *there is no exchange of thought either*. Thoughts are inseparably bound up with the subject. [...] By so-called exchange of thought with another being the subject only touches the outer wall of an automaton. This can hardly be called mutual understanding. [...] Only through the sensation of the other's soul sometimes a deeper approach is experienced. (CW I: 485)

This is a revelation of the potential for other people's autonomous existence and thus a way out for solipsism. Still, closing ourselves up into our innermost essence is argued to be the best way of avoiding suffering and anxiety, in expectation of the moment at which we will be definitively freed of such problems. Brouwer thus hopes for solipsism. This solipsism also underlies *his* mathematical intuitionism: what he wants is a mathematics that develops out of something inner, a-linguistically and free of applicative aims to the extent that he wants to keep the subject alone while the world is being built.

2. G. F. C. Griss

George François Cornelis Griss (1898–1953) was a student of Brouwer's who took a Ph.D. in Mathematics (under Weitzenboeck's supervision) and attended the Bolland lectures which introduced Hegelian philosophy into the Netherlands. In 1946, he wrote a slim philosophy volume – *Idealistische filosofie* – in which he expounded his *Weltanschauung* from the starting point of the original datum that consciousness comprehends when it has achieved fullness: the subject distinguishes itself from the object "but one has no meaning without the other."

The unification of memories is the "I" which is always accompanied by a certain set of experiences (the body) and other experiences linked to each other in a stable way (the bodies of other beings). We believe in the existence of other people since we see analogies between some of these bodies and our own. Furthermore, since we perceive analogies between their behavior and ours we presume that they possess a certain level of consciousness. Finally, when spiritual exchange with them occurs, we grasp their *full* consciousness.

The original datum can be considered from three different viewpoints:

1) philosophy (both comprehending and experiencing the subject--object bond) – the most complete viewpoint;

2) mysticism (experiencing the subject-object unity) – enabling people to feel the union between man and the totality of the world, between man and God;

3) mathematics (considering the object isolated from the subject as far as possible).

Griss defines this philosophy "empiricist idealism": idealist because it does not accept things-in-themselves and recognizes the importance of the *a priori*; empiricist because it also recognizes the importance of sensible experience. Philosophy neither depends on the validity of sciences nor derives from them[2] but is capable of founding ethics by taking the consequences of full awareness of the subject-object link to their culmination. Ethics establish that: 1) individuals grow only through contact with others; 2) every single man is responsible for all other men; 3) spiritual development does not take place without material growth.

As regards mathematics, it views the subject/object *distinction* as a scheme by which to generate entities (*in primis* natural num-

[2] The purpose of the sciences is describing phenomena, the purpose of philosophy is truth.

bers) but it cannot leave the subject/object link to one side, hence a mathematical entity exists (only) insofar as the person constructing it exists. *On this basis* Griss embraces intuitionism. It is clear that his *Weltanschauung* and Brouwer's are very different: they are diametrical opposites. Brouwer's is self-centered, considering other people's existence as something our destiny does not allow us to avoid entirely but one which we should try to side step as far as possible. Griss' viewpoint begins, like Brouwer's, from the self but cannot conceive of the self without the non-self and tasks individuals with responsibility for other people's well-being.

The Brouwer–Griss common ground is their shared acceptance of mathematical intuitionism. But here, too, Griss immediately marks his intuitionism out from Brouwer's by stressing that the bond linking all things implies that distinguishing clearly between them is impossible. Consequently, on one hand, the further mathematics pursues this, the more inexact it becomes and, on the other, it is this vagueness that lies at the very heart of infinite sequences. That is, "infinite" means that it is *not known* when it will end.

Furthermore, Griss feels the need to criticize the intuitionistic notion of negation (as a mental construction that can go no further):

> for example, we suppose that a fraction satisfies the equation $x^2 = 2$ and we find a contradiction because for each fraction that is substituted for x the first member differs from 2. Making the assumption that a proof is given, while that proof appears to be impossible, is inconsistent with the constructive and intuitive (evident) starting point, since the existence of a proof is identical to the fact that it was given. (1948: 71)

He proposes substituting the old definition of intuitionistic negation with a comparison between pre-existing entities and an acknowledgement that one has more properties than the other. For example: if we have two sets – *a* and *b* – differentiating a from b means showing which elements in *a* are not present in *b* or viceversa.

Brouwer responded in 1948 with a two-sheet paper (*Essentially negative properties*) arguing that we cannot prove that $\rho \neq 0$ is equivalent to "$r^\circ > 0$ or $\rho < {}^\circ 0$", that is, we cannot prove that negative property > is translatable into the expression "measurably larger (°>)" which is defined in merely affirmative terms, because $a^\circ > b$ means that $a - b > 2^{-n}$ for a suitable natural number *n*; ρ is a real number obtained as follows: α is a mathematical assertion that cannot be tested, i.e. no method has yet been identified with which to prove either its absurdity or the absurdity of its absurdity. The "creating subject" will build a sequence *a1*, *a2*, … by choosing:

- $a_n = 0$ as long as, in the course of choosing a_n, the creating subject has experienced neither the truth nor the absurdity of α;
- $a_{r+n} = 2^{-r}$ if between the a_{r-1} and a_r choices the creating subject has found proof that α is true;
- $a_{s+n} = -2^{-s}$ if between the a_{s-1} and a_s choices the creating subject has found proof that α is absurd.

The *a1*, *a2*, … sequence is positively convergent and ρ is its limit; $\rho \neq 0$, because its being equal to 0 would mean that it is always impossible to prove either the truth or the absurdity of α: $\neg(\alpha \vee \neg\alpha)$, which is equivalent to $\neg\alpha \wedge \neg\neg\alpha$, that is, a contradiction. In order to prove that we are not able to affirm $r^\circ > 0$ (that is $\rho > 2^{-n}$ for a suitable natural number *n*), Brouwer shows that we cannot even affirm $\rho > 0$: in this case $\rho < 0$ would be impossible. Hence the absurdity of the absurdity of α should already be a certainty, while it is not. In the same way, in order to prove that we cannot affirm $\rho < {}^\circ 0$, he shows that we cannot even affirm $\rho < 0$: in this case, $\rho > 0$ would be impossible. Hence the absurdity of α should be a certainty, while it is not.

Such an answer can be critiqued from multiple viewpoints. In this paragraph it is essential to stress that it does not counter Griss's criticism: the fact that an unacceptable notion (not equal) cannot be expressed through other acceptable ones (measurably larger/smaller) does not oblige us to accept it.

3. A. Heyting

It was precisely in Brouwer's 1948 paper that Arend Heyting – whose 1925 Ph.D. thesis supervisor was Brouwer – glimpses a *solipsistic turn* on Brouwer's part, arguing for it firstly in his 1967 Brouwer obituary: "As early as 1907 Brouwer connected his philosophy of mathematics with the philosophy of science. In later years he developed philosophical ideas which led him to solipsism; he explained his ideas in his lecture before the philosophical congress in 1948" (Heyting 1980 [hereafter CP]: 685).

Secondly, in 1968, in describing Brouwer's thought, Heyting affirms:

> Brouwer's line of thought may be characterized as solipsism [...]. Brouwer's philosophy of mathematics is intimately connected with his general philosophical ideas. Only the solipsistic turn of the latter is inessential for his foundations of mathematics. He has sometimes described mathematics as an activity of the mathematical community as a whole [...], his construction of intuitionistic mathematics is nothing more nor less than an investigation of the utmost limits which the intellect can attain in its self-unfolding. (CP: 697–698)

Finally, in 1978 (*History of the Foundations of Mathematics*) Heyting remarks that

> Brouwer sometimes defended solipsism (1948) and though he did not consistently apply this to mathematics he saw mathematics in the first place as conceived in the mind of an individual mathematician; the communication with colleagues comes afterwards and is always troubled by possible misunderstanding. (CP: 771)

The only possible inference from the Heyting quotes referred to above is that he intends solipsism as the belief that only one's own mind exists. Heyting also has no sympathy for the mathematical

"counterpart" (so to speak) of such solipsistic turns, i.e. the intuitionistic counter-examples to classical mathematics developed for the first time by referring at each instant to the knowledge of the so-called "creating[3] subject": "Brouwer introduced an idea for which the subject is of essential importance" (CP: 687), while "in intuitionistic mathematics the mathematical construction itself is relevant; the construction is objective in this sense that it is irrelevant which subject makes the construction" (CP: 687).

As early as his 1956 *Introduction to Intuitionism* Heyting discussed Brouwer's results in his chapter on "controversial subjects" and stated that he preferred van Dantzig approach to these results with van Dantzig having begun with a finite set of *mathematical deductions* and not from a creating subject[4] (Heyting 1956: 116).

In his unpublished writings (around 1978) Heyting examines the theme of solipsism in depth and to some extent rehabilitates it.

We will now move onto:

1) a focus on the concept of "existence" which is closely related to solipsism, by showing its various occurrences in Heyting's published and unpublished work;

2) illustrating the contents of these papers.

3.1. The notion of "existence" in Heyting's thought

Up to 1974 Heyting repeatedly examined the notion of existence but "only within mathematics." Firstly[5] he expressed his opposition to

[3] Heyting uses the word "creative" without feeling the need to explaining the reason for this change.

[4] In 1967 he refers to Kreisel's formalization of the introduction to a subject in intuitionistic set theory and expresses his dissatisfaction with it. In addition he cites van Danztig formalization of Brouwer's reasoning but adds that it was not a faithful rendering of Brouwer's thought (CP: 687).

[5] See his 1930 paper *Sur la logique intuitionniste* and also his 1968 *Philosophy of Mathematics* (*Wijsbegeerte van de Wiskunde*).

the potential for extending everyday realism around objects to all classical mathematical objects on the grounds that the latter refers to the infinite. Secondly,[6] he cited Poincare's meaning of mathematical existence as non-contradictoriness and saw it as bizarre. He also[7] underlined that the axiomatic viewpoint does not express an approach to the existence of objects: it is simply a matter of believing that *if* axioms are true *then* their consequence must also be true. In taking stock of the mathematical foundations status quo, in 1958 he further stressed[8] that classical mathematics is a magnificent assortment of formal, intuitionistic and Platonic parts (*Bestandteile*). The first two a mutual interdependence. That is, if a formalist seeks to develop mathematics using formal methods alone, then a relic of appeal to intuition remains, whilst if a intuitionist tries to develop mathematics by referring to number intuition alone, then a need for formulas will, at a certain point, make itself felt. Most mathematicians are Platonists: their proof methods (*Beweisverfahren*) are classical but not seen as pure formal inferences: they believe in the existence of a mathematical world. By contrast, intuitionists conceive existence as a mental construction even if they don't always accept the same "evidence." Heyting was conscious that full accordance can exist only around formalized notions but added that setting out a step by step table of evidence degrees (*eine Tafel von Evidenzstufen* – CP: 563) is possible on the basis of the requirements of each, starting with the only condition-free constructions (*bedingungslose*), i.e. those that can be seen (*ueberblickt*) immediately, such as 2 + 2 = 4. The impossible constructions identified (and critiqued) by Griss appear inside the table. At the top of the table is the general concept of mathematical proof (CP: 563–564). Heyting argued that Platonic

[6] See, for instance, his 1953 *Sur la tâche de la philosophie des mathématiques* and his 1956 *Intuitionism: An Introduction*.

[7] See also in *Sur la tâche de la philosophie des mathématiques* and *Intuitionism: An Introduction*.

[8] See his 1958a paper *Blick von der intuitionistischen Warte*.

mathematics could not be put at the top of such a table[9] of decreasing evidence because mathematical Platonism *is essentially different* from intuitionistic mathematics: accepting the former does not imply a new construction principle but rather a *philosophical thesis* whose meaning is not shared by all philosophers. Such a thesis was widespread among mathematicians until the turn of the 20th century but was powerfully undermined by the Hilbert and Brouwer schools and "seems to be increasing nowadays, under the influence of the successes of the semantic school, under the leadership of Tarski" (CP: 575).

On one hand he stressed[10] that the notion of mathematical truth (so carefully defined by Tarski) was meaningless to him, because mathematical entities cannot exist outside man: a construction is objective when it is irrelevant which subject built it. On the other hand he argued[11] that the intuitionistic meaning of existence made mathematics autonomous. Moreover, *the aim* of constructivism is[12] freeing mathematics from philosophical hypotheses. In 1974 (*Intuitionistic Views on the Nature of Mathematics*) he argues ironically that the only intuitionist philosophy consists in having no philosophy: we think in entities (and memorize these) and do not care how or why this is possible.

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[9] The final version of this table is the following (1962):

1) the highest grade of evidence is that of assertions like 2 + 2 = 4;

2) 1002 + 2 = 1004 belongs to a lower level: this is shown not by actual counting, but by reasoning which shows that, in general, (n + 2) + 2 = n + 4;

3) (n + 2) + 2 = n + 4: these general statements about natural numbers belong to the next grade;

4) the notion of the order type ω as it occurs in the definition of constructible ordinals;

5) the notion of negation;

6) the theory of quantification;

7) the introduction of infinitely proceeding sequences;

8) the notion of a species.

[10] See his 1958b paper *On Truth in Mathematics* (*Over waarheid in de wiskunde*).

[11] See 1967 *Informal Rigor and Intuitionism*.

[12] See his 1960 paper *Remarques sur le constructivisme*.

In his 1974 lecture *Science, Belief, Religion*, Heyting takes into account[13] the notion of existence, "in both science and everyday life." In this little known paper,[14] after presenting his definition of science, common to both natural and human sciences, Heyting treats the notion of existence through its four characteristics: "science introduces a structure into experience by using mathematics"; "the scientific structure includes more than direct experience"; "science aims at unity"; "science must have freedom to choose its subject and freedom of method" (Heyting 1974b: 200). He underlines that no natural science grasps the essence of nature: science is done on any subject with the awareness that it "must be content to imagine a suitable structure for the subject it is studying" (and only its successes can lead to judgements on a hypothesis's scientific value) (1974b: 202). This entails scientific openness: hypotheses can be rejected and replaced by more efficient ones. Heyting stresses that the purpose of science is not to obtain a mirror-image of reality. "Its goal is more modest" (Heyting 1974b: 202): it is the effectiveness of the image obtained. This is strictly linked to the notion of existence:

> In general, we say that an object exists when it is associated with an abstract entity of the structure in which we organize our experience and when it fulfills certain conditions which I can not specify here, but which correspond in outline to the syntax of the noun. […] The physicist is convinced of the existence of the electron as long as it works well in theory. For us to say that an object exists, it is necessary and sufficient that this object functions in a structure that is well adapted to the experience. (1974b: 202)

[13] For more information, see Franchella 2019.

[14] The French translation of the talk was *Science, croyance, foi* in S., Dockx (Ed.), *Science philosophie foi. Colloque de l'académie internationale de philosophie des sciences* (pp. 197–212). Paris: Beauchesne. The Dutch version of it appeared in 1977 in de Gids and was reprinted in A., Heyting (1980). *Collected Papers* (pp. 756–764).

Heyting notes that some authors attach a kind of existence to religious notions (such as "miracle," "god") which are very different from their meaning in everyday life and scientific contexts. For instance, Bishop Ian Ramsey specified that religious statement language contains certain words (called "models") relating to a realistic domain of reference accompanied by "qualificators" contrasting with models' realistic meanings. For example: "God is eternal (qualificator) father (model)." A qualificator tells us that we should not look for the meaning of the word concerned in a realistic domain.

> Qualificators have a role also in mathematics: What now of a regular polygon with an 'infinite number' of sides? [...] if we increased the sides without limit, but kept the area approximately constant, there may suddenly dawn on us at a certain point the outline of something quite different: a circle. At this point there would be evoked what we might call 'mathematical insight' – something akin to the disclosure of a characteristically religious situation. (Ramsey 1957: 61–69)

> In general, there are various situations in the sciences where such insight takes place: when "invariants" like points, force, mass are pointed out. What characterizes a religious context is that the qualificator leads us, through histories, i.e. developments of situations slowly approaching an ideal point, to the final insight. (Ramsey 1957: 61–69)

Heyting responds that:

> If the notion of God is a limiting notion in Ramsey's sense, then this notion does not meet the criterion, so I cannot say that God exists. We can extend the meaning of the word 'to exist' so as to make God exist, but then we give to this word a meaning very different from that which it has in everyday life, and it will be necessary to distinguish these two meanings. (Heyting 1974b: 208)

Furthermore, Heyting believes that considering miracles and gods as limit-notions to be approached means

> that we use images to express approximately our feelings (sentiments) attached to those notions. There isn't any reason to attribute those images an existence that we would never attribute to any other image. For instance, if I say that misery in Pakistan causes a move of compassion, I don't have the intention of considering this move as existent. (Heyting 1974b: 207–208)

It is clear that here Heyting, aiming to establish a consolidated distinction between what is scientific and what is not, strictly distinguishes between a concept of existence common to both science and everyday life as something which fits into a structuring of reality, and a religious concept of existence, "at the limits."

3.2. Heyting's unpublished papers

There are traces of profound and continuously reshaped reflections around solipsism in Heyting's unpublished papers, kept at Rijksarchief Utrecht and labelled "Group F. Philosophical notes." Most of the material was found in a red package named "Filosofie," held together by paper clips, ordinary packages and exercise books, or combined into a single group of papers quite obviously part of a single sequence. All of this was set out in Franchella (1995: 209–211) according to Troelstra 1989 Index of the *Nachlass*. Our attention here focuses on the following parts of this philosophical archive:

- F3–1/5 – five pages dated 22.06.1978 headed *Solipsism* on the front page;
- F6–1/13 – further sheets entitled *Over solipsism*. Anne Sjerp Troelstra hypothesized that these were written in preparation

for a talk on "filosofiedag" held at Centrale Interfaculteit, Erasmus Universiteit Rotterdam, 20.09.1980. F10 is a single leaf entitled *Uitbreiding van de betekenis van bestaan* (*Extension of the meaning of existence*), dated 21.07.1979;
- F7–1/12 – diary sheets;
- F8 – various sheets containing "numbered annotations";
- F11–1/8 – eight sheets, the first of which bears the heading *About the word "existence"* (*Over het woord bestaan*), presumably dating to later than 2.05.1980;
- F13–F18 – containing excerpts from authors quoted in his previous sheets (Dijksterhuis, Oliver, Gallie, Austin, Malcom, Ayer);
- F19 – three pages in English headed *Does an external world exist?* written for his daughter-in-law, Mrs. Heyting-Johnson, after a philosophical discussion in September 1979.

This may be a draft of Heyting's fundamental ideas followed by many disparate refinements and further reflections, stimulated by further references to ever wider literature on the subject. Heyting's purpose is to set down his views on solipsism, he also considers various other definitions of solipsism[15] and observes that philosophers often *fear* solipsism, and therefore misrepresent it. In fact, he argues that *affirming* one's belief in solipsism – because communication presupposes believing in the existence of other people – and providing a *theory* of solipsism – because theorizing always *presupposes* a reality against which it can be checked – are each impossible and self-contradictory. Therefore, solipsism can only be *expressed* in the form of a question: "Solipsism is a problem putting. Is the claim that an outside world exists more than the expression of a thought or a feeling? Can I be given something more than my own thoughts and feelings?" (F6.1). He adds that if a philosophical system considers a positive answer to a problem as one which is immediate, it is dogmatic. Philosophers should begin with *the fact* that a representation

[15] For more information, see Franchella 1995: 214–216.

of the outer world and other people exists and then *ask how* this representation of the outer world functions (*functionieert*) in spirit. The starting point is consciousness in its full indeterminacy, because we necessarily start with what we know primarily and certainly.

In this sense he rehabilitates solipsism as "prolegomena[16] to any philosophy" (F8.7). His toolbox consists of introspection (as usual within intuitionism!): "Does philosophy rest on empiry (empirie)? Yes, as far as introspection is a form of empiry" (F8.17). Introspection ultimately tells us that consciousness in its full indeterminacy means that the self has not yet distinguished itself from the non-self, and the connected beauty emotion (extasis), felt so strongly that nobody would be able to survive it if it lasted for a long time. Consequently, pure consciousness is difficult to describe: we refer to animals' consciousness because it is less intricate. Man implements certain forms of protection from pure consciousness consisting of ordering it internally. The first of these is "reason" which isolates the self. Groups of sensations are separated out from each other and distinguished from the group of sensations which the self consists of.[17] Heyting underlines that each distinction is relative, because everything is mixed up together in consciousness, but he considers the act of distinguishing as a means to survival. In his mathematical writings it was the basis of natural numbers, as it was with Griss: neither of them refered to time for this purpose, while Brouwer did.[18]

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[16] Heyting argues: "Solipsism as an experience (believing) can mean two things: the experience of the bee against the window-pane; the experience that my representation of the outer world and of other men as a spiritual enrichment" (F5, 31). The first is an unpleasant experience. On the contrary, the second is a good starting point also for philosophers.

[17] The self is active and programs the brain which controls the entire nervous system; the self is a problem-solver.

[18] We would stress here that in this sense in Heyting, too, (as in Griss) mathematics encompasses an original vagueness.

Once objects are identified, they need to be linked to one other. Temporalization comes before spatialization because many sensations with nothing to do with space exist: hunger, smell, etc. There is no absolute space *per se*, because there are no spatial representations without qualities, but rather sight and taste fit into a three-dimensional space (hence Euclidean geometry is a cultural product), and spatialization is always present in our bodily sensations. Furthermore, since we are aware that exchanging thoughts with other people enrich us,[19] this demonstrates that other people exist for us as *minds*. Hence, we attribute people with bodies like our own only subsequently, but in any case spirit and body cannot be separated. Heyting's primary approach to people is via the soul, while Griss's is via the body, though both Griss and Heyting begin with the content of our consciousness, and ultimately believe in the existence of human beings (both spirit and body).

It is only after we admit that other people exist that we can be sure that the outer world exists (i.e. we do not expect reality to change suddenly), because "certainty of the existence of the outer world means that anybody standing where we are would see what we are now seeing."

Brouwer's "exit" from consciousness to objects/living beings passed through time, mind and causality; space was excluded as *a priori* and his opinion of these steps was a negative one. Heyting finds concrete objects as soon as the I is separated from the non-I and time and space then follow, then people (having spiritual exchange with us and later recognized as bodies, too), and finally the outer world.

[19] Introspection is useful because it founds philosophy on empirical data. By contrast, psychology is not necessary to philosophy: it is a science, therefore it already presupposes the existence of an outer world.

Final remarks

In the 1974 meeting, where his aim was to establish a consolidated distinction between what is scientific and what is not, Heyting considered a unique concept of existence, common to both science and everyday life, as something which fits into a structured reality, one which is to be strictly distinguished from an existence understood as "at the limit," grasped by means of insight. In the years which followed, Heyting reconsidered the subject of "existence" in his unpublished papers noting that we construct a certain idea (model) of the world which generates certain expectations and this also takes place in accordance with the education we receive and thus depends on society. It allows us to distinguish between dream and reality (if something does not fit into the model it belongs to the dream world). Therefore, we believe in the existence of objects even if we cannot see them, both because they generate the simplest model of the world and because we learnt them at school and can trust others' perceptions: "Why am I convinced of the existence of Japan? Well, because I was told so at school, and because I imagine that some people perceive things there as I perceive my environment" (F7.11). Furthermore, it feels right when we think of our model. Still, there are doubts on the horizon relating to abstract objects:

> 3.10.78 Each of our abstract concepts begins with something simple and evident. So does "existence": First there are the objects of my direct environment, which exist; last there are stars and mesons. How many steps are there between these and how close the concept change as it passes from one to the other?
> God, real numbers and large cardinal numbers are at the top of such steps. At which step do the rules for the existential quantifier hold? (F5: 20)

He also transcribes some brief quotations from de Pater's book *Taalanalytische perspectieven op godsdiensten kunst* [Perspectives From an Analysis of Language Viewpoint on Religious Art]:

> Disclosure; passage to a limit
> The "I" can only be known by self-disclosure
> God can be known onIy by disclosure.
> Functions of proofs of God's existence:
> I. They are technical tools to bring about disclosure.
> 2. They give us clues to the logical behavior of the word "God". [The former coincides with the inspiration which comes from croyance] [also by using metaphors].
> A miracle is a miracle for the believer only; for others it is only an unexplained phenomenon.
> (F8.4)

This quote is very similar to that of Bishop Ramsey seen above, but Heyting neither stresses this fact nor comments on it.

He later (1980) offers a scale of existence:

> 1. the immediate environment. Within this, people.
> 2. representations of the environment and where I can go from there. The levels partially melt into each other. They overlap each other.
> 3. Memories of environments, where I have been before (here the concept of other men plays a role).
> 4. communications from other people (only at this stage the distinction dream/reality can be made clearly)
> 5. space-relations among represented environments
> 6. systematisation of these relations by means of maps and globes
> 7. fitting of all structures into a spatial generalization reaching till infinity.
> 8. Astronomy.
> 9. Microscopically small objects.

> 10. Theoretical physics. Particles of a second fraction exist. Here 'existence' has all become 'fitting into the theory'. (F11.5–6)

Immediately after this passage, entitled *Existence of abstract things* (*Bestaan van abstrakta*) he specifies that most people stop at the fifth step and that

> There is a second course of development of the concept 'existence' in addition to the first.
> I. Pain, falling in love, anger "exist".
> In accordance with the feeling of the outer world is the opposition to our will that this perceptions offer. Nevertheless, this existence is experienced as very different from that of the outer world.
> II. Gods and spirits, what we call superstitions. This belief is primitive science. One tries to explain phenomena which are not understood by analogy with everyday experience. (F11.5–6)

Our sensations also "exist": they oppose the same resistance as the outer world to our will. Still, they differ from our perception of the outer world both because they are neither spatialized nor shared by other people. Here we have *two* types of existence (one shared and another not shared by others), while the existence of gods and spirits seems to be excluded at our point of cultural development.

Whilst he frequently returns to the subject of "existence" and modifies some of his thoughts, in his final interview given to his daughter-in-law around the existence of the outer world, he confirms his positive belief in it with a feeling of security (defined "a careful *solipsistic way of expression* which is inevitable") and a feeling of resistance, against his will, adding that "This description of what the conviction of the existence of an external world means for me is certainly incomplete" (F19.1) and specifying that

> [...] the main respect in which this description is incomplete is that of the intersubjectivity. In my opinion the notion of other minds is more primitive than that of than outside world (I am afraid that very few people agree with this.) As long as he does not physically attack me (there are other exceptions) another man's mind is to me more important than his body. (Even if he threatens to attack me I try to change his mind, not his body).

Furthermore, he has *by nature* a strong conviction that other minds are similar to his own. He believes that not everybody shares his love for an individual woman but he is convinced that everybody will interpret his perception of a thing in the same way. Hence he is convinced that such perceptions are intersubjective. "In other words, I am convinced that such perceptions are in some sense intersubjective. This being granted, I might rewrite all the above, replacing everywhere 'I' by 'we'" (F19.3).

This allows Heyting to "open the window" that imprisoned Brouwer's intuitionism (so closely bound up with the uncommunicable inner world as to make it impossible to understand why the author himself would try to express it) and coherently enabled him to share it with other people.

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